

# Reverse mathematics of the finite downwards closed subsets of $\mathbb{N}^k$ ordered by inclusion

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## Abstract

We show that the well partial orderedness of the finite downwards closed subsets of  $\mathbb{N}^k$ , ordered by inclusion, is equivalent to the well foundedness of the ordinal  $\omega^{\omega^\omega}$ .

**Keywords:** reverse mathematics, well partial orderings, adjacent Ramsey

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In this note we prove the following theorem which was conjectured by Hatzikiriakou and Simpson in Remark 6.2 in [5].

**Definition 1** *We order  $k$ -tuples coordinatewise.*

**Theorem 2**  *$\text{RCA}_0$  proves that the following are equivalent:*

1.  $\omega^{\omega^\omega}$  is well founded,
2. For every  $k$ : the finite downwards closed subsets of  $\mathbb{N}^k$ , ordered by inclusion, are a well partial order.

The case of  $k = 2$  was shown by Hatzikiriakou and Simpson to be equivalent to the well-foundedness of  $\omega^\omega$ .

This note is also part of the attention, in reverse mathematics, for the strength of the well foundedness of the ordinals  $\omega^\omega$  and  $\omega^{\omega^\omega}$  (See, e.g.: [5, 6, 8]). Research on these

levels goes back to Simpson's work on the Hilbert and Robson basis theorems, in [9], or even further to Goodstein's work on his sequences, in [3].

We assume basic familiarity with reverse mathematics in  $\text{RCA}_0$  (II.1-II.3 in [10]) and treatment of ordinals less or equal to  $\omega^{\omega^\omega}$  (See, e.g. Definition 2.3 in [9] or Section II.3(a) in [4]).

Recall the basic definitions from order theory:

**Definition 3** *Given partial order  $(X, \leq)$ , we call a sequence  $x_0, x_1, \dots$  of elements from  $X$  bad if for all  $i < j$  we have  $x_i \not\leq x_j$ .*

**Definition 4** *A partial order is a well partial order (w.p.o.) if every bad sequence in the order is finite.*

**Definition 5** *A partial order is well founded if every strictly descending sequence in the order is finite.*

We will use the following principle from Friedman [2] for the upper bound:

**Definition 6 (Adjacent Ramsey for pairs)** *For every function  $C: \mathbb{N}^2 \rightarrow \mathbb{N}^r$  there exist  $a < b < c$  with  $C(a, b) \leq C(b, c)$ .*

**Theorem 7**  $\text{RCA}_0$  *proves that the following are equivalent:*

1.  $\omega^{\omega^\omega}$  *is well-founded,*
2. *the adjacent Ramsey theorem for pairs.*

*Proof:* Theorem 3 in Section 4 of [7].

□

*Proof of Theorem 2 (1)  $\rightarrow$  (2):* Take, for a contradiction, an infinite bad sequence  $G_0, G_1, G_3, \dots$  with:

$$G_i = \{m_{i,0} \dots, m_{i,n_i}\}$$

Define:

$$C(i, j) = m_{i,l}$$

where  $l \leq n_i$  is the smallest such that  $\forall p \leq n_j. m_{i,l} \not\leq m_{j,p}$ . By adjacent Ramsey there exist  $a < b < c$  such that  $C(a, b) \leq C(b, c)$ , contradiction.

□

**Definition 8** *A downwards closed subset  $X$  is generated by  $G$  if*

$$X = \{m \in \mathbb{N}^k : \exists m' \in G. m \leq m'\}.$$

Every finitely generated set is also finite (upper bound given by the generators).

For finite sets we say  $G \leq H$  if  $X \leq Y$ , where  $X$  and  $Y$  are the respective generated sets.

Notice that  $G \not\leq H$  if and only if there exists  $m \in G$  with  $\forall m' \in H. m \not\leq m'$ .

*Proof of Theorem 2 (2)  $\rightarrow$  (1):* For  $\beta = \omega^k \cdot b_0 + \dots + \omega^0 \cdot b_k < \omega^{k+1}$ , take  $h(\beta) = (b_0, \dots, b_k) \in \mathbb{N}^{k+1}$ . We have the following property:  $h(\beta) \leq h(\beta') \rightarrow \beta \leq \beta'$ .

For  $\alpha =_{\text{CNF}} \omega^{\beta_0} \cdot a_0 + \dots + \omega^{\beta_n} \cdot a_n < \omega^{\omega^{k+1}}$ , define:

$$f(\alpha) = \{(i, a_i) \wedge h(\beta_i) : i \leq n\}.$$

Notice that  $f(\alpha)$  is an antichain in  $\mathbb{N}^{k+3}$ .

Assume, for a contradiction, that  $\omega^{\omega^{k+1}} > \alpha_0 > \alpha_1 > \dots$  is an infinite sequence and let  $i < j$  be such that  $f(\alpha_i) \leq f(\alpha_j)$  by well-partial-orderedness. Denote:

$$\alpha_i =_{\text{CNF}} \omega^{\beta_{i,0}} \cdot a_{i,0} + \dots + \omega^{\beta_{i,n_i}} \cdot a_{i,n_i},$$

$$\alpha_j =_{\text{CNF}} \omega^{\beta_{j,0}} \cdot a_{j,0} + \dots + \omega^{\beta_{j,n_j}} \cdot a_{j,n_j}.$$

Let  $l$  be the smallest such that  $(l, a_{i,l}) \wedge h(\beta_{i,l}) \not\leq (l, a_{j,l}) \wedge h(\beta_{j,l})$ , such  $l$  exists because otherwise  $\alpha_i \leq \alpha_j$ .

Let  $q > l$  be the smallest such that  $(l, a_{i,l}) \wedge h(\beta_{i,l}) \leq (q, a_{j,q}) \wedge h(\beta_{j,q})$ , such  $q$  exists because of  $f(\alpha_i) \leq f(\alpha_j)$ .

By the properties of the Cantor Normal Forms, we have the following for all  $p \geq l$ :

$$\omega^{\beta_{j,l}} > \omega^{\beta_{j,q}} \geq \omega^{\beta_{i,l}} \geq \omega^{\beta_{i,p}}.$$

Hence, by  $\omega^{\beta_{j,l}}$  being closed under ordinal addition,  $\alpha_i \leq \alpha_j$ , contradiction.

□

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